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Some inverse problems and fractional calculus

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Several inverse problems in physics are modeled in terms of nonlinear integral equations of the Abel type. The main aim of this survey article is to show a method with fractional calculus is commonly effective for proving the global existence of solutions to the integral equations.

A typical inverse problem reduced to a nonlinear integral equation of the Abel type is the problem to determine a restoring force of the Newtonian equation such that the equation has a prescribed half-period as a function of the half-amplitude of of the solution:

Problem 1 Determine a nonlinearity g of an equation $\frac{d^2u}{dt^2}+g(u)=0$ so that, for each $x\in(0,r]$, a solution $u(t)$ of the equation with the stationary (maximal) value x has a half-period $T(x)$ that is a prescribed function of x.

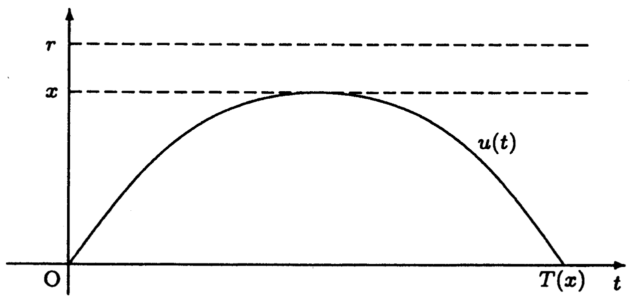

Figure 1: Problem 1.

As is easily verified by a standard discussion (see, e.g., [16]), Problem 1 is reduced to the integral equation:

$$\sqrt{2}\int_0^x \frac{dy}{\sqrt{\int_y^x g(u)du}}=T(x),\quad 0\leq x\leq r. \tag{1}$$

Here $r>0$, $T(x)$ is a prescribed, positive function, and we seek a solution g that is continuous on the interval $[0,r]$, positive on the interval $(0,r]$. The uniqueness of a continuous solution g of (1) was established by Opial [10]. The existence result for Problem 1 was firstly obtained by Urabe [15, 16], which showed that (1) admits a solution g if r is small, under the assumption that T has a Lipschitz continuous derivative. This local

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existence result was improved by Alfawicka [1], which showed that (1) admits a solution g if r is small, under the assumption that T itself is Lipschitz continuous. However whether the solution exists globally (in the sense that r is arbitrary) had been a well-known open problem in the field of inverse problems, until the author [9] has closed this open problem recently by proving the following global version of the local existence result due to Alfawicka.

Theorem 2 ([9]) *Given a Lipschitz continuous, positive function T on the interval $[0,r]$, there exists a (unique) solution g of (1) that is continuous on $[0,r]$ and positive on $(0,r]$.*

We give an example indicating the meaning of Theorem 2:

Example 3 Let $-\infty < \alpha < 1$, let $G(x)$ be the inverse function of the incomplete beta function

$$x(t) := \int_0^t s^{-\frac{1}{2}}(1-s)^{-\alpha} ds,$$

and let $T(x)$ be a function defined by

$$T(x) = \sqrt{2}\,\pi F\left(\alpha, \frac{1}{2}, 1; G(x)\right)$$

with the Gauss hypergeometric function $F(\alpha, \beta, \gamma; t)$. Then the solution g of (1) is given by

$$g(x) = G(x)^{\frac{1}{2}}(1-G(x))^{\alpha}, \quad 0 \le x < x_0 := B\left(\frac{1}{2}, 1-\alpha\right).$$

One can verify (see [9]) that g satisfies (2) with the function $T(x)$ for $0 \le x < x_0$.

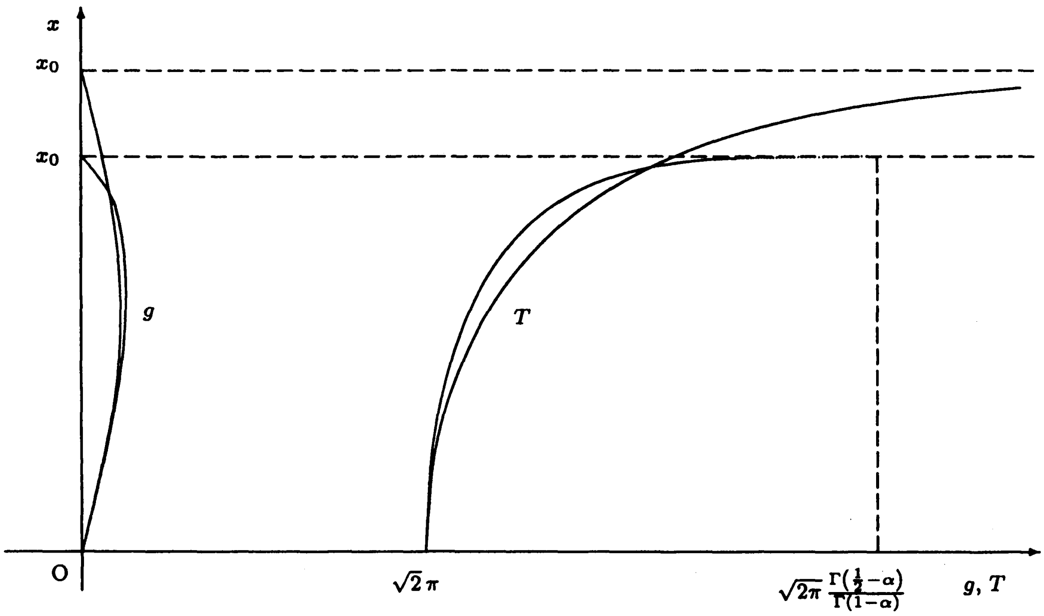

Figure 2: $g(x)$ and $T(x)$ for $\alpha \in (0,1)$

In the case $\frac{1}{2} \le \alpha < 1$ (see the upper part in Fig 2), the prescribed function $T(x)$ is increasing monotonically from $\sqrt{2}\pi$ $(= T(0))$ to $+\infty$ as x moves from 0 to x_0. Then Theorem 2 is applicable to $T(x)$ for any positive $r < x_0$ because $T(x)$ satisfies the conditions in

Theorem 2 when we take r in $(0, x_0)$. The lower bound $\alpha = \frac{1}{2}$ in this case is corresponding to the simple pendulum since $g(x) = \frac{1}{2}\sin x$ and $T(x) = \sqrt{2}\,\pi\, F\left(\frac{1}{2}, \frac{1}{2}, 1; \sin^2\frac{x}{2}\right)$ for $\alpha = \frac{1}{2}$. In the case $0 < \alpha < \frac{1}{2}$ (see the lower part in Fig 2), the prescribed function $T(x)$ is increasing monotonically from $\sqrt{2}\pi\,(= T(0))$ to $T(x_0) = \sqrt{2\pi}\Gamma(1/2 - \alpha)/\Gamma(1-\alpha)$ as x moves from 0 to x_0. However, in this case, $T(x)$ is not Lipschitz continuous at x_0. Hence, as well as in the case $\frac{1}{2} \le \alpha < 1$, Theorem 2 is applicable by taking r in $(0, x_0)$. Notice that we are not able to take r as x_0 because $g(x) = 0$ at $x = x_0$. The lower bound $\alpha = 0$ in this case is corresponding to a spring with a homogeneous elasticity, since $g(x) = \frac{1}{2}x$ and $T(x) \equiv \sqrt{2}\,\pi$ for $\alpha = 0$. In the case $0\alpha < 0$ the prescribed function $T(x)$ is decreasing monotonically from $\sqrt{2}\pi\,(= T(0))$ to $T(x_0) = \sqrt{2\pi}\Gamma(1/2 - \alpha)/\Gamma(1-\alpha)$ as x moves from 0 to x_0. However, either in this case, $T(x)$ is not Lipschitz continuous at x_0; $g(x)$ is going to $+\infty$ as $x \to x_0$. Theorem 2 is also applicable by taking r in $(0, x_0)$.

The proof of Theorem 2 is crafted by an appropriate combination of of fractional calculus and successive approximations in [9]. Here we explain only a core of it. By letting $x(t)$ be the inverse function of $t = \int_0^x g(u)du$, equation (1) is rewritten as

$$\frac{\sqrt{2}}{2\pi}\int_0^t \frac{T(x(s))}{\sqrt{t-s}}ds = x(t). \tag{2}$$

Applying a method of successive approximations we can get a solution $x(t)$ of this equation reaching r (see Figure 3). So our task is to show that $x(t)$ is monotonically increasing, because g is the derivative of the inverse function of $x(t)$.

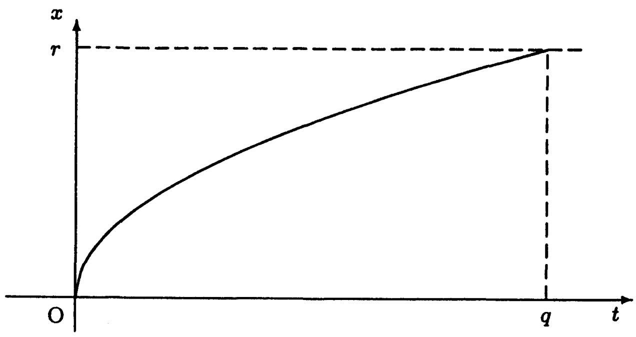

Figure 3: Solution $x(t)$

Proposition 4 *Let T be a Lipschitz continuous function on an interval containing* 0 *and assume that* $T(0) > 0$. *If a continuous function $x(t)$ defined on some bounded, closed interval* $[0, q]$ *satisfies* (2) *then $x(t)$ is differentiable and the derivative $x'(t)$ is positive on* $(0, q]$.

The proof of this proposition is based upon the fractional calculus associated with a Riemann-Liouville integral operator I^δ defined by

$$(I^\delta\phi)(t) = \frac{1}{\Gamma(\delta)}\int_0^t \frac{\phi(s)}{(t-s)^{1-\delta}}ds,$$

(Γ is the Gamma function) and and a corresponding differential operator D^δ defined by $D^\delta = DI^{1-\delta}$, where D is a standard differential operator $D = d/dt$. Generally speaking, the Riemann-Liouville integral operators improve the Hölder continuity of functions by their order δ, while the Riemann-Liouville differential operators D^δ have the converse character (see Samko, Kilbas and Marichev [12]). We use this mapping property of the Riemann-Liouville operators within the framework of a Hölder space, which is a a modified version of the result due to Hardy and Littlewood [2].

First we note that equation (2) is written as

$$I^{\frac{1}{2}} \frac{T \circ x}{\sqrt{2\pi}} = x.$$

Applying the operator $D^{\frac{1}{2}}$ to this equality we get

$$\frac{T \circ x}{\sqrt{2\pi}} = I^{\frac{1}{2}} x',$$

and, in turn, letting ε be a small, positive number and applying the operator $D^{\frac{1}{2}}$ to this resulting equation, we arrive at

$$D^{1-\varepsilon} \frac{T \circ x}{\sqrt{2\pi}} = D^{\frac{1}{2}-\varepsilon} x'.$$

If the set $\{t \in (0, q] \mid x'(t) = 0\}$ were not empty then we can show that $(D^{\frac{1}{2}-\varepsilon})(a) \leq -\rho < 0$ at the smallest point in the set, where ρ is a positive number independent of ε. On the other hand we can get

$$\lim_{\varepsilon \to 0} \left(D^{1-\varepsilon} \frac{T \circ x}{\sqrt{2\pi}} \right)(a) = 0$$

The assumption that T is Lipschitz continuous is used essentially at this stage. In this way we have got a contradiction. This is an outline of the proof of Proposition Proposition 4. We wish to point out that this proposition is of independent interest in the field of integral equations, apart from Problem 1.

Problem 1 is interpreted as a part of an inverse bifurcation problem (for several inverse bifurcation problems, see Iwasaki and Kamimura [3, 4], Kamimura [7], Shibata [13]). As is well-known in a general bifurcation (see, e.g., Rabinowitz [11]), if f is continuous with $f(0) > 0$ then the first bifurcating branch of the nonlinear eigenvalue problem

$$\begin{cases} u'' + \lambda u f(u) = 0 \quad \text{on} \quad (0,1), \\ u(0) = u(1) = 0, \\ u \neq 0 \quad \text{on} \quad (0,1). \end{cases} \tag{3}$$

bifurcates at the point $\left(\frac{\pi^2}{f(0)}, 0\right)$ from the trivial solution. By the condition that $u \neq 0$ on the interval $(0,1)$, each solution u of (4) is positive or negative in the interval. Hence the solution has its maximum value or minimum value at the middle point $\frac{1}{2}$ in the interval (see Figure 4). By means of the value h, as a projection into $(0,\infty) \times I$ of the first bifurcating branch of (4), we define a set $\Gamma(f)$ in $(0,\infty) \times I$ by

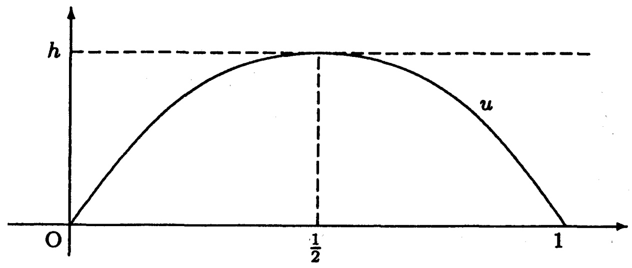

Figure 4: Solution of equation (3).

$$\Gamma(f) := \{(\lambda, h) \in (0,\infty) \times I \,|\, \exists u \in C^2[0,1] \text{ satisfying (3) and } u(\tfrac{1}{2}) = h\}, \tag{4}$$

where I is a bounded, closed interval containing 0. Let us confine ourselves in the case where $f(u) > 0$ on I in what follows. It is easy o see that the set $\Gamma(f)$ is represented as $\Gamma(f) = \{(\lambda(h), h) : h \in I \setminus \{0\}\}$ via a positive function $\lambda(h)$ defined by

$$\lambda(h) = 2\left(\int_0^1 \frac{dt}{\sqrt{\int_t^1 sf(hs)ds}}\right)^2. \tag{5}$$

In this way we may define a correspondence

$$\mathcal{B} \,:\, f(u) \mapsto \lambda(h),$$

which we refer to as the bifurcation transform. It should be noted that the trivial function $f \equiv f(0)$ is mapped by the bifurcation transform $\mathcal{B}$ to the trivial bifurcation $\lambda(h) \equiv \frac{\pi^2}{f(0)}$, which is corresponding to the linear case.

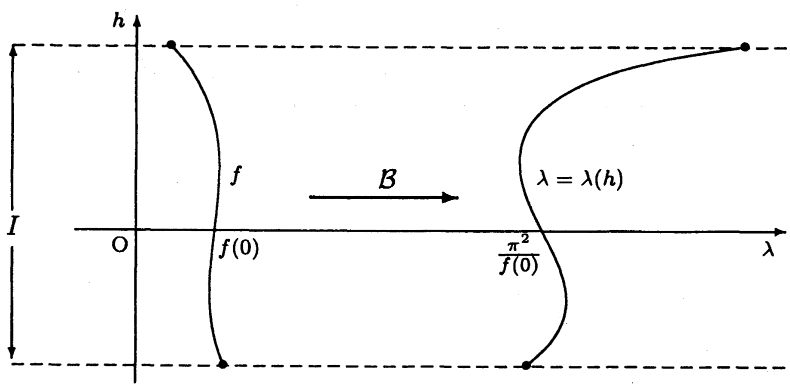

Figure 5: Bifurcation Transform.

Our inverse bifurcation problem is to ask whether f can be recovered from $\lambda(h)$, which consists of four questions:

Problem 5

1. (Existence) Given a positive function λ on I, does there exist a positive function f on I such that $\mathcal{B}f = \lambda$?
2. (Uniqueness) Is f unique for each λ ?
3. (Stability) Does f depend on λ continuously ?
4. (Reconstruction) Can one reconstruct f from λ ?

An answer to questions 1 and 2 of Problem 5 is obtained as an immediate rewriting of Theorem 2:

Theorem 6 *Given a Lipschitz continuous, positive function λ on I, there exists a unique continuous function f on I such that $\mathcal{B}f = \lambda$. The function f is obtained in a constructive way.*

Notice that Theorem 6 is a global result in the field of inverse bifurcation problems. In addition, since a method of proof to Theorem 2 is constructive (recall that the solution of $x = x(t)$ to equation (2) is obtained by successive approximations), we can get a general strategy for question 4, namely, for reconstruction of the nonlinearity f.

When one considers the bifurcation transform $\mathcal{B}$ as a map, the space of Lipschitz continuous functions is not so suitable, because the inverse image of the space via the transform is not well-characterized. So, instead of the space of Lipschitz continuous functions, we introduce the following Hölder-like spaces with $\alpha \in (0, 1]$:

$$\mathcal{C}^{0,\alpha}(I)_1 := \Bigg\{\phi \in C(I) \;:\; ||\phi||_{0,\alpha,1} := \sup_{h\in I\setminus\{0\}} \frac{|\phi(h)|}{|h|} + \sup_{h,k\in I\setminus\{0\},h\neq k} \frac{||h|^{\alpha-1}\phi(h) - |k|^{\alpha-1}\phi(k)|}{|h-k|^{\alpha}} < \infty\Bigg\},$$

$$\mathcal{C}^{1,\alpha}(I)_1 := \left\{\psi \in C(I) \cap C^1(I \setminus \{0\}) \;:\; \psi(0) = 0,\; h\psi'(h) \in \mathcal{C}^{0,\alpha}(I)_1\right\}.$$

Equipped with the norms $||\phi|| := ||\phi||_{0,\alpha,1}$ and $||\psi|| := ||h\psi'(h)||_{0,\alpha,1}$ respectively, the spaces $\mathcal{C}^{0,\alpha}(I)_1$ and $\mathcal{C}^{1,\alpha}(I)_1$ are Banach spaces. A suitable choice of metric spaces setting for the transform $\mathcal{B}$ is the combination of

$$\mathcal{M}^{0,\alpha}(I)_1 := \left\{f \in C_+(I) \;:\; f(h) - f(0) \in \mathcal{C}^{0,\alpha}(I)_1\right\}$$

and

$$\mathcal{M}^{1,\alpha-\frac{1}{2}}(I)_1 := \left\{\lambda \in C_+(I) \;:\; \lambda(h) - \lambda(0) \in \mathcal{C}^{1,\alpha-\frac{1}{2}}(I)_1\right\},$$

with $\alpha \in (\frac{1}{2}, 1)$, where $C_+(I)$ denotes the set of continuous, positive functions and the metrics of $\mathcal{M}^{0,\alpha}(I)_1$ and $\mathcal{M}^{1,\alpha-\frac{1}{2}}(I)_1$ are defined by

$$d(f_1, f_2) := |f_1(0) - f_2(0)| + ||(f_1(h) - f_1(0)) - (f_2(h) - f_2(0))||_{0,\alpha,1}$$

and

$$d(\lambda_1, \lambda_2) := |\lambda_1(0) - \lambda_2(0)| + ||(\lambda_1(h) - \lambda_1(0)) - (\lambda_2(h) - \lambda_2(0))||_{1,\alpha-\frac{1}{2},1}$$

respectively. With the aid of these metric spaces, we have an answer to questions 1–3 of Problem 5:

Theorem 7 *Let $\frac{1}{2} < \alpha < 1$. Then $\mathcal{B}$ is a homeomorphism of $\mathcal{M}^{0,\alpha}(I)_1$ onto $\mathcal{M}^{1,\alpha-\frac{1}{2}}(I)_1$.*

Theorem 7 implies that, for each first bifurcating branch $\lambda \in \mathcal{M}^{1,\alpha-\frac{1}{2}}(I)_1$, there exists a unique nonlinearity f in $\mathcal{M}^{0,\alpha}(I)_1$ realizing the first bifurcating branch (the answer to question 1-2), and in addition, that the correspondence $\lambda \mapsto f$ is continuous with respect to the topology of $\mathcal{M}^{0,\alpha}(I)_1$ and $\mathcal{M}^{1,\alpha-\frac{1}{2}}(I)_1$ induced from the metrics of these spaces (the answer to quention 3), provided that $\alpha \in (\frac{1}{2}, 1)$. Since the space $\mathcal{M}^{0,1}(I)_1$ is no other than that of Lipschitz continuous functions, Theorem 6 can be regarded as the limit case of Theorem 7 as $\alpha \to \frac{1}{2} + 0$, though the conclusion in Theorem 7 seems to break down when $\alpha = \frac{1}{2}$.

We omit the proof of Theorem 7, which is rather long and technical, deviating from the aim of this article.

In this article we have discussed a method with fractional calculus that is applicable in proving the global existence of solutions to integral equations related with inverse problems. The method is also available for a heat conductivity determination problem:

Problem 8 *Given functions $f(t), g(t)$, determine $a(t)$ so that the parabolic system*

$$\begin{cases} u_t = a(t)u_{xx}, & 0 < x < \infty,\ 0 < t < T; \\ u(x,0) = 0, & 0 \le x < \infty; \\ u(0,t) = f(t), & 0 \le t < T; \\ -a(t)u_x(0,t) = g(t), & 0 < t < T \end{cases} \tag{6}$$

admits a bounded, solution $u(x,t)$.

This inverse problem was studied by Jones [5, 6], Suzuki [14], Kamimura [8]. The original idea of fractional calculus applicable to inverse problems can be found in [8],

References

[1] B. Alfawicka, *Inverse problems connected with periods of oscillations described by $\ddot{x} + g(x) = 0$*, Ann. Polon. Math. **44** (1984), 297–308.

[2] G.H. Hardy and J.E. Littlewood, *Some properties of fractional integrals. I*, Math. Zeitschrift, **27** (1928), 564–606.

[3] K. Iwasaki and Y. Kamimura, *An inverse bifurcation problem and an integral equation of the Abel type*, Inverse Problems **13** (1997), 1015–1031.

[4] K. Iwasaki and Y. Kamimura, *Inverse bifurcation problem, singular Wiener-Hopf equations, and mathematical models in ecology*, J. Math. Biol. **43** (2001), 101–143.

[5] B. F. Jones Jr. , *The determination of a coefficient in a parabolic differential equation, Part I, existence and uniqueness*, J. Math. Mech. **11** (1962), 907–918

[6] B. F. Jones Jr. , *Various methods for finding unknown coefficients in parabolic differential equations*, Comm. Pure. Appl. Math. **16** (1963), 33-44.

[7] Y. Kamimura, *An inverse problem in bifurcation theory*, J. Differential Equations **106** (1993), 10–26.

[8] Y. Kamimura, *Conductivity identification in the heat equation by the heat flux*, J. Math. Anal. Appl. **235** (1999), 192–216.

[9] Y. Kamimura, *Global existence of a restoring force realizing a prescrived half-period*, J. Differential Equations, in press.

[10] Z. Opial, *Sur les périodes des solutions de l'équation differentielle* $x'' + g(x) = 0$. Ann. Polon. Math. **10** (1961), 49–72.

[11] P.H. Rabinowitz, *Nonlinear Sturm-Liouville problem for second order ordinary differential equations*, Comm. Pure Appl. Math. **23** (1970), 939–961.

[12] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives, Gordon and Breach, Switzerland, 1993.

[13] T. Shibata, *L^q-inverse spectral problems for semilinear Sturm-Liouville problems*, Nonlinear Analysis **69** (2008), 3601–3609.

[14] T. Suzuki, Mathematical Theory of Applied Inverse Problems: Sophia Kokyuroku in Math. Vol. 33, Sophia University, Tokyo, 1991.

[15] M. Urabe, *The potential force yielding a periodic motion whose period is an arbitrary continuous function of the amplitude of the velocity*, Arch. Rational Mech. Anal. **11** (1962), 27–33.

[16] M. Urabe, Nonlinear Autonomous Oscillations, Academic, New York, 1967.